Modular symmetries and identities for rank-type partition statistics

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Background and motivation

- A *partition* π of n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of unrestricted partitions of n.
- Ramanujan's classical congruences are

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+6) \equiv 0 \pmod{11}.$$

- Dyson's *rank* and the Andrews-Garvan *crank* organize partitions into congruence classes that explain these congruences.

Rank : Largest part minus number of parts. **Crank** : Largest part if there are no ones; otherwise number of parts larger than the number of ones minus the number of ones.

- Mock theta functions, that were realized by Zwegers in 2000 as real analytic parts of weight $\frac{1}{2}$ harmonic weak Maass forms, arise naturally from rank-type generating functions at roots of unity. This motivates the study of modular completions, identities, and p-dissections.

Generating functions for the rank and crank

Let $\zeta_p = e^{2\pi i/p}$ and $q = e^{2\pi i\tau}$, $\tau \in \mathscr{H}$. The partition rank and crank generating functions are

$$R(z,q) = \sum_{n\geq 0}\sum_{m} N(m,n) z^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq;q)_n (z^{-1}q;q)_n}.$$

Here $N(m,n)$ denotes the number of partitions of n with rank equal to m.

$$C(z,q) = \sum_{n\geq 0}\sum_{m} M(m,n) z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}.$$

Here $M(m,n)$ denotes the number of partitions of n with crank equal to m.

The partition rank : modular structure

- Bringmann-Ono (Ann. Math., 2010) : Build on Zwegers' work to prove that $R(\zeta_p, q)$ is the holomorphic part of a weak Maass form of weight $\frac{1}{2}$ on $\Gamma_1(576p^4)$.
- Bringmann-Ono-Rhoades (JAMS, 2008) : Use the previous result to prove that the rank sum modulo p is a weakly holomorphic modular form of weight $\frac{1}{2}$ on $\Gamma_1(576p^6)$.
- Garvan's improvement (TAMS, 2019) : Adjusted rank function $\eta(p^2z)R_p(\zeta_p, z)$ is weakly holomorphic of weight 1 on $\Gamma_0(p^2) \cap \Gamma_1(p)$, where

$$R_p(\zeta_p,z) = q^{-1/24}R(\zeta_p,q) - \left(\frac{12}{p}\right)\sum_{a=1}^{(p-1)/2}(-1)^a\left(\zeta_p^{3a+(p+1)/2} + \zeta_p^{-3a-(p+1)/2} + \zeta_p^{3a+(p-1)/2} - \zeta_p^{-3a-(p-1)/2}\right)q^{\frac{a}{2}(p-3a)-\frac{p^2}{24}}\Phi_{p,a}(q^p),$$

$$\text{where } \Phi_{p,a}(q) = \begin{cases} \sum_{n=0}^{\infty} \dfrac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & 0 < 6a < p, \\ -1 + \sum_{n=0}^{\infty} \dfrac{q^{pn^2}}{(q^a;q^p)_{n+1}(q^{p-a};q^p)_n}, & p < 6a < 3p. \end{cases}$$

Symmetry observations

A 7-dissection of $R(\zeta_7, q)$ from Ramanujan's Lost Notebook Part III can be written in generalized eta-products as

$$\begin{aligned} &R(\zeta_7,q) + (\zeta_7+\zeta_7^6-2)\phi_{7,1}(q^7) + (-\zeta_7^2+\zeta_7^3+\zeta_7^4-\zeta_7^5)q^{-1}\phi_{7,2}(q^7) + (1+2\zeta_7^2+\zeta_7^3+\zeta_7^4+2\zeta_7^5)q^{-5}\phi_{7,3}(q^7) \\ &= (-1+\zeta_7+\zeta_7^6)\frac{\eta(49z)\eta_{7,3}(7z)}{\eta_{7,1}(7z)\eta_{7,2}(7z)} + \frac{\eta(49z)}{\eta_{7,1}(7z)} + (\zeta_7+\zeta_7^6)\frac{\eta(49z)\eta_{7,2}(7z)}{\eta_{7,1}(7z)\eta_{7,3}(7z)} + (1+\zeta_7^2+\zeta_7^5)\frac{\eta(49z)}{\eta_{7,2}(7z)} - (\zeta_7^2+\zeta_7^5)\frac{\eta(49z)}{\eta_{7,3}(7z)} \\ &\quad - (1+\zeta_7^3+\zeta_7^4)\frac{\eta(49z)\eta_{7,1}(7z)}{\eta_{7,2}(7z)\eta_{7,3}(7z)}. \end{aligned}$$

Multiplying by $\eta(49z)$ and replacing q by $q^{1/7}$ gives the 7-dissection of $\eta(p^2z)R_p(\zeta_p, z)$ for $p = 7$.

$$\mathcal{K}_{7,0} = 0, \qquad \mathcal{K}_{7,1} = -(1+\zeta_7^3+\zeta_7^4)\frac{\eta(7z)^2\eta_{7,1}(z)}{\eta_{7,2}(z)\eta_{7,3}(z)}, \qquad \mathcal{K}_{7,2} = (-1+\zeta_7+\zeta_7^6)\frac{\eta(7z)^2\eta_{7,3}(z)}{\eta_{7,1}(z)\eta_{7,2}(z)},$$
$$\mathcal{K}_{7,3} = \frac{\eta(7z)^2}{\eta_{7,1}(z)}, \qquad \mathcal{K}_{7,4} = (\zeta_7+\zeta_7^6)\frac{\eta(7z)^2\eta_{7,2}(z)}{\eta_{7,1}(z)\eta_{7,3}(z)}, \qquad \mathcal{K}_{7,5} = (1+\zeta_7^2+\zeta_7^5)\frac{\eta(7z)^2}{\eta_{7,2}(z)},$$
$$\mathcal{K}_{7,6} = -(\zeta_7^2+\zeta_7^5)\frac{\eta(7z)^2}{\eta_{7,3}(z)},$$

m	$-24m \pmod 7$	behavior
$1, 2, 4$	quadratic residues	one orbit
$3, 5, 6$	non-residues	another orbit
0	zero	fixed type

A symmetry phenomenon for the partition rank

Let $\mathcal{K}_{p,m}(\zeta_p^d; z)$ be the adjusted rank dissection element attached to the m-th part of the p-dissection of $\eta(p^2z)R_p(\zeta_p, z)$.

Theorem (Garvan-Sarma, Ramanujan J., 2024). Suppose $p > 3$ prime, $0 \leq m \leq p-1$, $1 \leq d \leq p-1$, and

$$A = \begin{pmatrix} a & k \\ p & d \end{pmatrix} \in \Gamma_0(p), \qquad 1 \leq a, d \leq p-1.$$

Then

$$\mathcal{K}_{p,m}(\zeta_p, z)\,\big|\,[A]_1 = \frac{\sin(\pi/p)}{\sin(d\pi/p)}(-1)^{d+1}\zeta_p^{mak}\mathcal{K}_{p,\overline{ma^2}}(\zeta_p^d, z).$$

Proofs employ transformation results for theta functions, their Mordell integrals, Klein forms and Zwegers' Maass forms.

Corollary. The identities for $\mathcal{K}_{p,m}(\zeta_p, z)$ in terms of generalized eta functions are completely determined by three particular ones namely

$$\mathcal{K}_{p,0}(\zeta_p, z),\ \mathcal{K}_{p,m^+}(\zeta_p, z),\ \mathcal{K}_{p,m^-}(\zeta_p, z),$$

where $\left(\frac{-24m^+}{p}\right) = 1$, $\left(\frac{-24m^-}{p}\right) = -1$.

The overpartition rank : completion and modularity

An *overpartition* is a generalization of partition in which the first occurrence of a part may be overlined. Its Dyson rank is also the largest part minus number of parts. Let

$$\mathcal{O}(z,q) = \sum_{n\geq 0}\sum_{m} \overline{N}(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n(n+1)/2}}{(zq;q)_n (z^{-1}q;q)_n}.$$

Here $\overline{N}(m,n)$ denotes the number of overpartitions of n with rank equal to m. The overpartition rank has a non-holomorphic obstruction. A Lambert-series correction cancels this obstruction.

$$\Phi_{p,k}(q) = q^{k-k^2/p}\frac{(q^{2p};q^{2p})_\infty}{(q^p;q^p)_\infty^2}\sum_{n\in\mathbb{Z}}(-1)^n\frac{q^{pn(n+k)}}{1-q^{pn+k}}.$$

Theorem (Garvan-Sarma, 2025). Let $p > 3$ prime and $1 \leq k \leq \frac{1}{2}(p-1)$. Then $\frac{\eta(p^2\tau)^2}{\eta(2p^2\tau)}\mathfrak{O}_p(\zeta_p, q)$ for

$$\mathfrak{O}_R(\zeta_p, q) := \mathcal{O}(\zeta_p; q) - \frac{2(1-\zeta_p)^{\frac{1}{2}(p-1)}}{1+\zeta_p}\sum_{k=1}^{}(-1)^k\left(\zeta_p^{-2k} - \zeta_p^{2k}\right)\Phi_{p,k}(q^p)$$

is a weakly holomorphic modular form of weight 1 on $\Gamma_0(2p^2) \cap \Gamma_1(p)$.

Symmetry for overpartition rank and crank

Define $\mathcal{OK}_{p,m}^{(R)}(\zeta_p^\ell; z)$ and $\mathcal{OK}_{p,m}^{(C)}(\zeta_p^\ell; z)$ as the (adjusted) dissection elements for the overpartition rank and crank.

Theorem (Garvan-Sarma, 2025). Suppose $p \geq 3$ prime, $0 \leq m \leq p-1$, $1 \leq \ell \leq p-1$, and $A = \begin{pmatrix} a & k \\ 2p & d \end{pmatrix} \in \Gamma_0(2p)$. Then

$$\mathcal{OK}_{p,m}^{(R)}(\zeta_p^\ell, z)\,\big|\,[A]_1 = \frac{1-\zeta_p^\ell}{1-\zeta_p^{\ell d}}\frac{1+\zeta_p^{\ell d}}{1+\zeta_p^\ell}\zeta_p^{mak}\mathcal{OK}_{p,ma^2}^{(R)}(\zeta_p^{\ell d}, z),$$

$$\mathcal{OK}_{p,m}^{(C)}(\zeta_p^\ell, z)\,\big|\,[A]_1 = \frac{\sin(\ell\pi/p)}{\sin(\ell d\pi/p)}\zeta_p^{mak}\mathcal{OK}_{p,ma^2}^{(C)}(\zeta_p^{\ell d}, z).$$

Algorithmic-automorphic program

The method combines explicit computation using results for automorphic forms.

1. Find a Jacobi/generalized eta-product basis for the relevant group using Sage.
2. Lift rank/crank differences to a weight where poles can be controlled.
3. Use cusp orders and the valence formula to prove identities.
4. Use group actions (the symmetry result) to transfer identities between dissection elements.

Target : Finding explicit p-dissections for overpartition rank and crank modulo p in terms of generalized eta-products, using Maple and Sage.

A glimpse of experimentation in Maple

```
CUSP FORMS ON GAMMA1(22) WITH WEIGHT 3
M=CuspForms(Gamma1(22),3, prec=150);M
b=M.basis();
file = open('C22_3.txt', 'w')
file.write(str(b))
file.close()
```

SageMath 9.1 Shell

```
sage: load("C22_3.sage")
Cuspidal subspace of dimension 20 of Modular Forms space of dimension 40 for Congruence Subgroup
 Gamma1(22) of weight 3 over Rational Field
sage:
```

```
> read "C22_3.txt":
> CD01B:=series(oCD11(0,1,0)*jac2series(JAC(11,22,infinity),200),q,150):
> FL:=[seq(series(CD01B*C22_3[k],q,200),k=1..20)]:
> BFL:=[op(FL),op(M22_4)]:
> symBFL:=[seq(_C01*_FM[k],k=1..20),seq(_M[j],j=1..55)]:
> rel:=findhom(BFL,q,1,0);
                                  # of terms , 96
                        -----RELATIONS----of order---, 1
rel := {16 X8 - 96 X9 + 8 X10 + 25 X11 + 64 X12 - 48 X13 + 56 X14 - 23 X15 - 40 X16 - 144 X17 + 64 X18 + 24 X19 - 112 X20 - X21 + 7 X22 - 24 X23 + 19 X24 + 22 X25 - 41 X26 - 36 X27 + 126 X28 - 68 X29 - 68 X30 + 166 X31 - 90 X32 - 54 X33 (135
  + 148 X34 - 188 X35 - 71 X36 + 468 X37 - 347 X38 - 220 X39 + 456 X40 - 124 X41 - 443 X42 + 276 X43 - 132 X44 + 48 X45 + 784 X46 - 411 X47 - 914 X48 + 1112 X49 - 136 X50 - 1040 X51 + 1039 X52 + 16 X53 + 468 X54 + 182 X55 + X1
  - 8 X2 + 23 X3 - 24 X4 - X5 - 8 X6 + 48 X7}
```

Figure: In search of a suitable Jacobi product basis for $\Gamma_1(22)$ to express the crank differences in terms of generalized eta products

A rank mod 11 identity for the partition rank

An identity for a quadratic-residue case ($m = 1$) is

$$\mathcal{K}_{11,1}(\zeta, z) = q^{1/11}(q^{11};q^{11})_\infty\left(\sum_{n=1}^{\infty}\left(\sum_{k=0}^{10}N(k,11,11n-4)\zeta^k\right)q^n + (\zeta+\zeta^{10}-\zeta^9-\zeta^2)q^{-1}\Phi_{11,5}(q)\right)$$
$$= \frac{f_{11,5}(z)}{f_{11,1}(z)}\sum_{r=1}^{5}c_{11,r}j(11,\pi_r(\vec{n}_1),z) + \frac{f_{11,4}(z)}{f_{11,5}(z)}\sum_{r=1}^{5}d_{11,r}j(11,\pi_r(\vec{n}_1),z),$$

where $\zeta = \zeta_{11}$, $\vec{n}_1 = (15,-4,-2,-3,-2,-2)$,

$$j(p,\vec{n},z) = \eta(pz)^{n_0}\prod_{k=1}^{(p-1)/2}f_{p,k}(z)^{n_k},$$

and

$$\begin{aligned} c_{11,1} &= d_{11,1} = d_{11,2} = d_{11,4} = 0, \\ c_{11,2} &= 5\zeta^9+\zeta^8+4\zeta^7+2\zeta^6+2\zeta^5+4\zeta^4+\zeta^3+5\zeta^2+5, \\ c_{11,3} &= -(5\zeta^9+3\zeta^7+2\zeta^6+2\zeta^5+3\zeta^4+5\zeta^2+1), \\ c_{11,4} &= \zeta^9-\zeta^8-\zeta^7-\zeta^4-\zeta^3+\zeta^2-2, \\ c_{11,5} &= -(6\zeta^9+2\zeta^8+3\zeta^7+5\zeta^6+5\zeta^5+3\zeta^4+2\zeta^3+6\zeta^2+5), \\ d_{11,3} &= \zeta^9+\zeta^8+\zeta^6+\zeta^5+\zeta^3+\zeta^2+1, \\ d_{11,5} &= -(2\zeta^9+\zeta^8+\zeta^7+\zeta^6+\zeta^5+\zeta^4+\zeta^3+2\zeta^2+1). \end{aligned}$$